On decomposable universal graphs (Model theoretic aspects of the notion of independence and dimension)

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Abstract

R.Diestel et al. proved that whether there is a universal graph in the class of all countable $\Gamma-$free graphs or not, where Γ is the class of subdivisions of K_n. In this note, we try to construct a generic structure for some subclass of them.

1. Existence of decomposable universal graphs

We recall some definitions at first. In this note, we define graph structures as follows.

Definition 1 Let the language $L=\{R(x, y)\}$ and $R(x, y)$ be a binary relation symbol.
An $R-$structure G is said to be a *graph* if
$R(x, y)$ is symmetric, $G\models\forall x\forall y[R(x, y)\longrightarrow R(y, x)]$,
$R(x, y)$ is irreflexive, $G\models\forall x[\neg R(x, x)]$.

Definition 2 Let $\mathcal{G}$ be a class of countable graphs.
A member G of $\mathcal{G}$ is called (*strongly*) *universal in* $\mathcal{G}$ if every $G'\in \mathcal{G}$ is isomorphic to some (induced) subgraph of G.
An infinite graph G is *ultrahomogeneous* if every isomorphism between finite induced subgraphs of G is extended to an automorphism of G.

By this definition, universal graphs may have no saturation.
There are some results on the existence of universal graphs for classes $\mathcal{G}$ characterized by the notions, subdivision and minor of graphs. We recall the definitions of them.

Definition 3 A *subdivision* of a graph X, denoted by TX, is any graph arising from X by replacing its edges with independent paths of length ≥ 1.

Definition 4 Let G be a graph and $V(G)$ be its vertex set. And let X be another graph and $\{V_x : x\in V(X)\}$ is a partition of $V(G)$ into connected subsets such that ;

for any two vertices x, $y \in V(X)$, there is a $V_x - V_y$ edge in G if and only if x and y are adjacent in X.

In this situation, we say that there exists a *contractive homomorphism fro m G onto X* and denote $G = HX$.

And we call X is a *minor* of G if G has a subgraph G' such that $G' = HX$.

Theorem 5 *(R.Diestel, R.Halin and W.Vogler [2])*

For Γ a class of countable graphs, we denote $\mathcal{G}(\Gamma)$ the class of all countable graphs that do not contain any subgraph isomorphic to a member of Γ.

Then $\mathcal{G}(TK_4) = \mathcal{G}(HK_4)$ has a strongly universal element, and for any n with $5 \leq n \leq \aleph_0$, $\mathcal{G}(TK_n) = \mathcal{G}(HK_n)$ has no universal element.

It is known that 2−connected graphs are constructed from a cycle by successively adding paths. Some refined argument of it is used to show the existence of universal graph in 2−connected members of $\mathcal{G}(TK_4) = \mathcal{G}(HK_4)$.

We recall some definitions and lemma. In the next lemma, we denote by $\mathcal{G}$ the class $\mathcal{G}(TK_4) = \mathcal{G}(HK_4)$ and by $\mathcal{G}^2$ the class of all 2−connected graphs in $\mathcal{G}$.

Definition 6 Let G be a graph and $\mathcal{P}$ a set of finite paths in G. Call another set $L = L(\mathcal{P})$ of finite paths in G a *labelling* of $\mathcal{P}$ if each path in L is contained in some path of $\mathcal{P}$.

A labelling L is *admissible* if $T \subset T'$ or $T' \subset T$ whenever T, $T' \in L$ are not edge-disjoint.

Let H be a graph and $G \subset H$, and $\mathcal{P}$ an admissible labelled set of finite paths in G. We call H an *admissible extension* of G with respect to $\mathcal{P}$ if there exists an admissible labelled set $\mathcal{P}_{\mathcal{H}}$ of independent $G - G$ paths in H such that

$$H = G \cup \bigcup_{P \in \mathcal{P}_{\mathcal{H}}} P$$

and the endvertices of each $P \in \mathcal{P}_{\mathcal{H}}$ coincide with the endvertices of some $T \in L(\mathcal{P})$.

Lemma 7 *Every $G \in \mathcal{G}^2$ can be expressed as $G = \bigcup_{i=1}^{\infty} G_i$ with $G_i \in \mathcal{G}^2$ for $i = 2, 3, \cdots$ in such a way that there exists a set $\mathcal{P}_0$ and $\mathcal{P}_i$ of independent $G_i - G_i$ paths in G for $i = 1, 2, \cdots$ such that*

1) $G_1 \cong K_2$,
2) $G_{i+1} = G_i \cup \bigcup_{P \in \mathcal{P}_i} P$,
3) *G_{i+1} is an admissible extension of G_i with respect to $\mathcal{P}_{i-1}$.*

All members of $\mathcal{G}^2$ are constructed as above. By means of this property, they construct a universal graph G^2 of $\mathcal{G}^2$ first. And for every vertex of G^2, infinitely many copies of G^2 are pasted randomly. So they realize a universal graph in $\mathcal{G}$.

On the other hand, in the case $5 \leq n < \aleph_0$, $\mathcal{G}(TK_n) = \mathcal{G}(HK_n)$ has uncountably many members. They negate the existence of universal graph in relation to the decomposability of it.

And in the case $n = \aleph_0$, they also reach the negation by some argument of combinatorics.

The argument of graph decomposition are related to Graph Minor Theorem. And many characterizations are obtained. In this note, we recall the definition of graph decomposition developed by R.Halin and R.Diestel.

Definition 8 Let G be a graph, $\sigma > 0$ an ordinal, and let B_λ be an induced subgraph of G for every $\lambda < \sigma$.

The family $F = (B_\lambda)_{\lambda<\sigma}$ is called a *simplicial tree – decomposition of G (into primes)* if the following four conditions hold :

(S1) $G = \bigcup_{\lambda<\sigma} B_\lambda$,

(S2) $(\bigcup_{\lambda<\mu} B_\lambda) \cap B_\mu = S_\mu$ is a complete graph for each μ $(0 < \mu < \sigma)$,

(S3) no S_μ contains B_μ or any other B_λ $(0 \leq \lambda < \mu < \sigma)$.

(S4) each S_μ is contained in B_λ for some $\lambda < \mu < \sigma$.

((S5) each B_λ is not separated by a simplex.)

There is a result by R.Halin.

Theorem 9 *Every graph not containing an infinite simplex (complete graph) admits a simplicial decomposition into primes.*

2. Decomposable generic graphs

By the last theorem, we can consider that the argument in the previous section is characterization of decomposable graphs. Thus they construct a decomposable universal graph. And the strongly universal graph G of $\mathcal{G}(TK_4) = \mathcal{G}(HK_4)$ has homogeneity to some degree, but G is not ultrahomogeneous.

In model theory many important examples of generic structure have been constructed. In general, they have strong homogeneity and saturation. And most of them are graph structures constructed by amalgamation property. In this section, we try to characterize some decomposable generic graphs.

We begin with the definitions of amalgamation property and Fraïssé limit (generic structure).

In the following, for sets $A \subset B$, we denote $B \setminus A = \{b \in B : b \notin A\}$.

Definition 10 Let L be a language and let $\mathbf{K}$ be a class of finite L-structures.

We say that $\mathbf{K}$ has *amalgamation property* if for any $A \subset B_1 \in \mathbf{K}$ and $A \subset B_2 \in \mathbf{K}$, there are $C \in \mathbf{K}$ and $B_1{}' \subset C$, and $B_2{}' \subset C$ such that $A \subset C$

and $B_1' \cong_A B_1$, and $B_2' \cong_A B_2$.

In particular, we say that $\mathbf{K}$ has *free amalgamation property* if for any $A \subset B_1 \in \mathbf{K}$ and $A \subset B_2 \in \mathbf{K}$, there are $C = B_1 \otimes_A B_2 \in \mathbf{K}$ and $B_1' \subset C$, and $B_2' \subset C$ such that $A \subset C$ and $B_1' \cong_A B_1$, and $B_2' \cong_A B_2$ satisfying that there is no relation between $B_1' \setminus A$ and $B_2' \setminus A$.

Theorem 11 *Let L be a language and $\mathbf{K}$ be a class of (isomorphism types of) finite L-structures.*

Suppose that $\emptyset \in \mathbf{K}$ and $\mathbf{K}$ is closed under substructures, and $\mathbf{K}$ has amalgamation property,

then there is a countable $L-$structure M with the following properties ;

1. *Any finite $X \subset M$ is a member of $\mathbf{K}$,*
2. *If $A \subset B \in \mathbf{K}$ and $A \subset M$, then there is a copy $B' \subset M$ such that $B' \cong_A B$.*

A countable $L-$structure having the properties 1 and 2 above is called a Fraïssé Limit (generic structure) of $\mathbf{K}$.

It is easily checked that $\mathcal{G}(TK_4) = \mathcal{G}(HK_4)$ has no amalgamation property.

Example 12 *Let A be a graph with vertices $\{a_i : i < 9\}$ such that ;*

$\{a_0, a_2, a_3\}$ and $\{a_1, a_3, a_4\}$ are triangles and $\{a_i : 2 \leq i \leq 8\}$ is a cycle, and there is no other edge in A,

and let B and C be extensions of A such that ;

B is the extension of A with an $A - A$ path of length 3 whose endvertices are $\{a_2, a_4\}$, and

C is also the extension of A with an $A-A$ path of length 4 whose endvertices are $\{a_3, a_5\}$.

Then there is no amalgam of B and C over A in $\mathcal{G}(TK_4) = \mathcal{G}(HK_4)$.

In this section, we try to construct a 2−connected generic graph for some class $\mathbf{K}$ of finite graphs. We settle notions to fix the class $\mathbf{K}$.

Definition 13 For a graph G and $\mathcal{P}$ a set of finite paths in G, we define a *labelling* of $\mathcal{P}$ as Definition 6.

Let H be a graph and $G \subset H$, and $\mathcal{P}$ a labelled set of finite paths in G. We call H an *extension* of G with respect to $\mathcal{P}$ if there exists a labelled set $\mathcal{P}_{\mathcal{H}}$ of independent $G - G$ paths in H such that

$$H = G \cup \bigcup_{P \in \mathcal{P}_{\mathcal{H}}} P$$

and the endvertices of each $P \in \mathcal{P}_{\mathcal{H}}$ coincide with the endvertices of some $T \in L(\mathcal{P})$.

Definition 14 A finite graph G is *constructible with respect to labels* if G can be expressed as $G = \bigcup_{i<n} G_i$ with G_i for $i < n$ in such a way that there exists a set $\mathcal{P}_0$ and $\mathcal{P}_i$ of independent $G_i - G_i$ paths in G for $i < n-1$ such that

1) $G_0 \cong K_2$,
2) $G_{i+1} = G_i \cup \bigcup_{P \in \mathcal{P}_i} P$,
3) G_{i+1} is an extension of G_i with respect to $\mathcal{P}_{i-1}$.

In the definition above, we take $\mathcal{P}_i$ maximally at each stage, as the set of chordless cycles with G_i.

Here we define a set of labelling restrictively.

Definition 15 Let a finite graph G be constructible with respect to labels such that $G = \bigcup_{i<n} G_i$, and $\mathcal{P}_i$ is independent $G_i - G_i$ paths in G for $i < n-1$.

We define a labelling $L(\mathcal{P}_{i-1})$ as the set of all those subpaths T of some $P' \in \mathcal{P}_{i-1}$ that form a cycle together with some $P \in \mathcal{P}_i$. For $P \in \mathcal{P}_i$, we take its labelling T with the minimal length.

We say that G has a labelling with *length* n if every labelling T of G (in all stages of construction) has the length at most n.

Let $P \in \mathcal{P}_i$ be a path. We say that the labelling $T(P)$ is *compatible* if there are independent paths $P_k \in \mathcal{P}_{j_k}$ for $k < 2$ and $j_k < i$ such that $T(P)$ and P_k are not edge-disjoint for $k < 2$ (that is, there is no single $P' \in \mathcal{P}_j$ such that $T(P) \in P'$ for some $j < i$).

Now we determine a class $\mathbf{K}$ as a rather easy case at first.

Definition 16 Let $\mathbf{K}^2$ be the class of finite graphs G satisfying that ;

1) G is constructible with respect to labels with length 2, whichever edge in G is chosen as G_0, and

2) G has no edges contained in different compatible labels (at the same stage in the construction).

Remark 17 *$\mathbf{K}^2$ contains all finite members of $\mathcal{G}^2$ with length 2. And the free amalgam $B \otimes_A C$ of Example 12 is in $\mathbf{K}^2$. Moreover $\mathbf{K}^2 \subset \mathcal{G}(TK_5) = \mathcal{G}(HK_5)$.*

Conjecture 18 *Let $\mathbf{K}^2$ be the class of finite graphs satisfying the conditions as above.*

Then $\mathbf{K}^2$ has free amalgamation property.

I have written the proof, but I need some time to check that all cases of factors in the amalgamation are considered.

3. Further problems

Problem 19 *Are there other classes of finite graphs which have amalgamation property, such as, the class of finite graphs which is constructible with respect to labels with length n?*

Problem 20 *Can we characterize decomposable generic graphs by predimension or dimension of generic structures? More generally, can we classify decomposable graphs by stability theoretic notions?*

Apology I found a mistake in the proof of *Corollary* 24 in my note "*Some remark on graph decomposition*", RIMS Kokyuroku No.1938.

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